Positive solutions for Large Random Linear Systems

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joint work with Pierre Bizeul

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Motivation

- Feasibility and stability in ecological networks.

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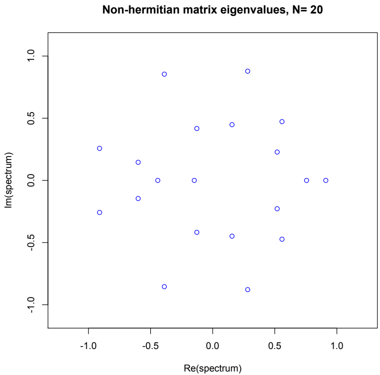

Figure: Distribution of $A_N/\sqrt{N}$'s eigenvalues

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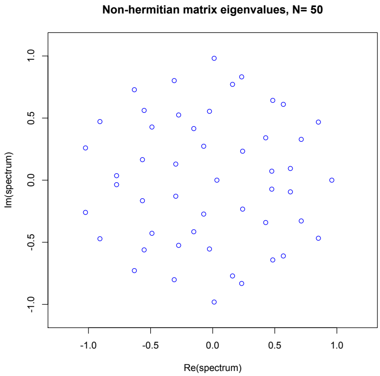

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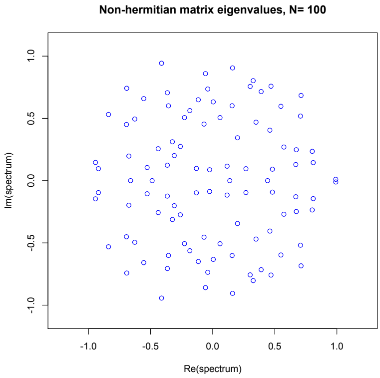

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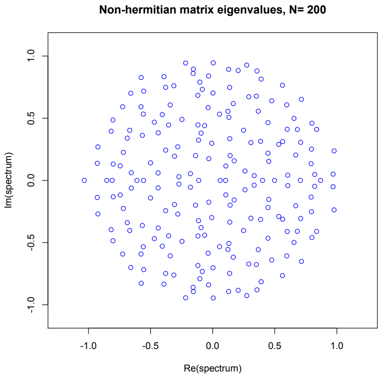

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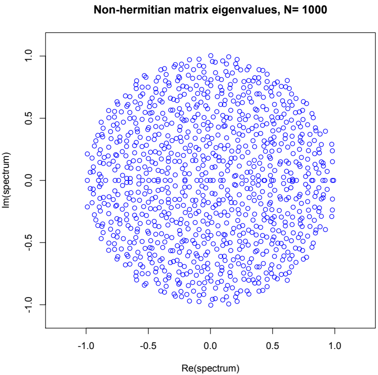

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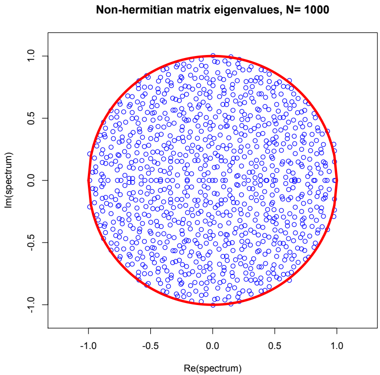

Figure: The circular law (in red)

Theorem: The Circular Law (Ginibre, Metha, Girko, Tao & Vu, etc.)

The spectrum of $\frac{A}{\sqrt{N}}$ converges to **the uniform probability on the disc**

Existence of a solution .. with no positive components

- From the spectrum confinement property,

$$\boldsymbol{x} = \left(I - \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\right)^{-1} \mathbf{1} \quad \text{exists for } \boldsymbol{\alpha} > \mathbf{1}$$

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- As a consequence

$$\mathbb{P}\left\{ \inf_{k \in [N]} x_k > 0 \right\} \quad \sim \quad \mathbb{P}\left\{x_k > 0\right\}^N \quad \xrightarrow[N \to \infty]{} \quad 0\,.$$

$\Rightarrow$ no positive solutions

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- If

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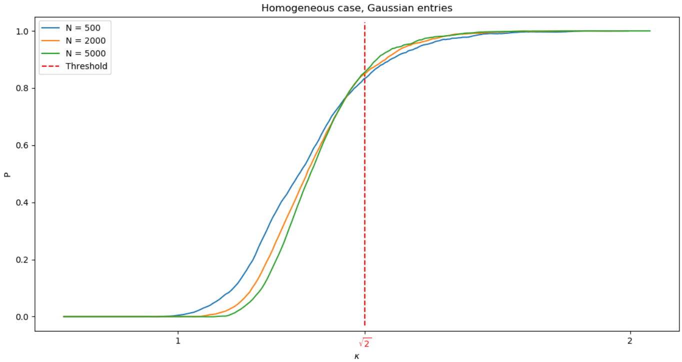
Phase transition (gaussian case)



- We plot the frequency (over 500 trials) of positive solutions for the linear system

$$\boldsymbol{x} = \mathbf{1} + \frac{1}{\kappa\sqrt{\log(N)}}\frac{A}{\sqrt{N}}\boldsymbol{x}$$

  as a function of the normalization parameter κ.
- As expected, we observe threshold phenomenon around the critical value $\kappa = \sqrt{2}$.

A heuristics for the critical scaling

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2. **Notice that**

$$Z_k \sim \mathcal{N}(0,1) \quad \text{i.i.d.} \qquad \text{and} \qquad \min_{k \in [N]} Z_k \sim -\sqrt{2\log(N)}$$

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Thank you for your attention!