Convergence and Dynamical Behavior of the ADAM Algorithm for Non Convex Stochastic Optimization

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Optimization in Deep Learning

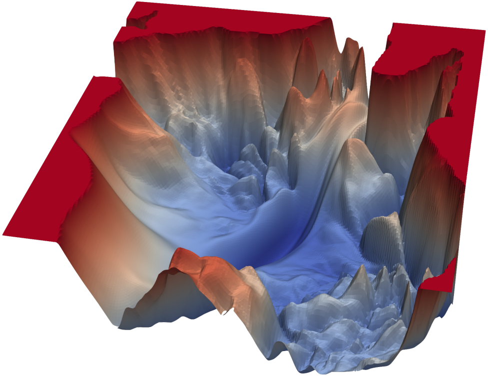


Figure 1: Visualization of a loss landscape (VGG-56 on CIFAR-10)
https://www.cs.umd.edu/ tomg/projects/landscapes/

Li et al., Visualizing the Loss Landscape of Neural Nets, NeurIPS 2018

Problem statement

Problem

$$\min_x F(x) := \mathbb{E}(f(x,\xi)) \quad \text{w.r.t.} \quad x \in \mathbb{R}^d$$

Assumptions

- $f(\,.\,,\xi)$: **nonconvex** differentiable function
- regularity assumptions on f (smoothness, coercivity of F, etc.)
- $(\xi_n : n \geq 1)$: iid copies of r.v ξ revealed online

ADAM : an adaptive algorithm
[Kingma and Ba, 2015]

- Regime : **constant step size** $\gamma > 0$.

Algorithm 1 ADAM $(\gamma, \alpha, \beta, \varepsilon)$

$x_0 \in \mathbb{R}^d, m_0 = 0,\ v_0 = 0,\ \gamma > 0,\ \varepsilon > 0,\ (\alpha, \beta) \in [0,1)^2$.

for $n \geq 1$ **do**

$\quad m_n = \alpha m_{n-1} + (1-\alpha)\nabla f(x_{n-1}, \xi_n)$

$\quad v_n = \beta v_{n-1} + (1-\beta)\nabla f(x_{n-1}, \xi_n)^2$

$\quad \hat{m}_n = \frac{m_n}{1-\alpha^n}$

$\quad \hat{v}_n = \frac{v_n}{1-\beta^n}$

$\quad x_n = x_{n-1} - \frac{\gamma}{\varepsilon + \sqrt{\hat{v}_n}} \hat{m}_n$

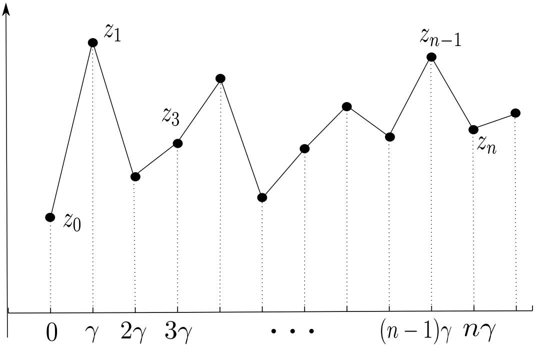
$\quad x_n = x_{n-1} - \gamma \nabla f(x_{n-1}, \xi_n)$ (SGD for comparison)

end for

From Discrete to Continuous Time

The ODE Method [Ljung, 1977, Kushner and Yin, 2003]

$\mathsf{z}^\gamma(t)$ interpolated from $z_n^\gamma = (x_n^\gamma, m_n^\gamma, v_n^\gamma)$



Continuous Time System

similar approach to [Su, Boyd and Candès, 2016]

Non autonomous ODE

If $z(t) = (x(t), m(t), v(t))$,

$$\dot{z}(t) = h(t, z(t)) \tag{ODE}$$

Theorem (Convergence)

$$\lim_{t\to\infty} \mathrm{d}(x(t), \nabla F^{-1}(\{0\})) = 0\,.$$

$$c_1(t)\,\ddot{x}(t) + c_2(t)\,\dot{x}(t) + \nabla F(x(t)) = 0\,,$$

- 2nd vs 1st order: acceleration (even if oscillations).
- Escaping local traps (saddle points)

Long run convergence of the ADAM iterates

- No a.s convergence : regime $n \to \infty$ then $\gamma \to 0$

Theorem (ergodic convergence of the ADAM iterates)

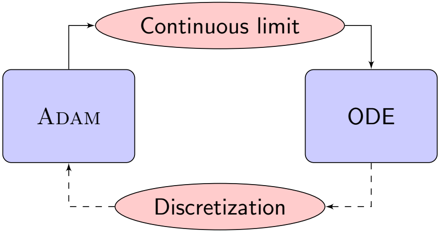
Let $x_0 \in \mathbb{R}^d$, $\gamma > 0$, $(z_n^\gamma : n \in \mathbb{N})$, $z_0^\gamma = (x_0, 0, 0)$. Under the same assumptions and :

- **Stability assumption:** $\sup_{n,\gamma} \mathbb{E}\|z_n^\gamma\| < \infty$.

Then, for all $\delta > 0$,

$$\lim_{\gamma \downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(\mathrm{d}(x_n^\gamma, \nabla F^{-1}(\{0\})) > \delta) = 0\,. \tag{1}$$

Thank you for your attention



For more details: submitted article, available on arXiv.

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